7.6 Mutual inductance

Two circuits, or loops, C_1 and C_2 are fixed in position relative to one another (Fig. 7.19). By some means, such as a battery and a variable resistance, a controllable current I_1 is caused to flow in circuit C_1. Let

$\mathbf{B}_1(x, y, z)$ be the magnetic field that would exist if the current in C_1 remained constant at the value I_1, and let Φ_{21} denote the flux of $\mathbf{B}_1$ through the circuit C_2. Thus

$$\Phi_{21} = \int_{S_2} \mathbf{B}_1 \cdot d\mathbf{a}_2, \tag{7.36}$$

where S_2 is a surface spanning the loop C_2. With the shape and relative position of the two circuits fixed, Φ_{21} will be proportional to I_1:

$$\frac{\Phi_{21}}{I_1} = \text{constant} \equiv M_{21}. \tag{7.37}$$

Suppose now that I_1 changes with time, but *slowly enough* so that the field $\mathbf{B}_1$ at any point in the vicinity of C_2 is related to the current I_1 in C_1 (at the same instant of time) in the same way as it would be related for a steady current. (To see why such a restriction is necessary, imagine that C_1 and C_2 are 10 meters apart and we cause the current in C_1 to double in value in 10 nanoseconds!) The flux Φ_{21} will change in proportion as I_1 changes. There will be an electromotive force induced in circuit C_2, of magnitude

$$\mathcal{E}_{21} = -\frac{d\Phi_{21}}{dt} \quad \Longrightarrow \quad \mathcal{E}_{21} = -M_{21}\frac{dI_1}{dt}. \tag{7.38}$$

In Gaussian units there is a factor of c in the denominator here. But we can define a new constant $M'_{21} \equiv M_{21}/c$ so that the relation between $\mathcal{E}_{21}$ and dI_1/dt remains of the same form.

We call the constant M_{21} the coefficient of *mutual inductance*. Its value is determined by the geometry of our arrangement of loops. The units will of course depend on our choice of units for $\mathcal{E}$, I, and t. In SI

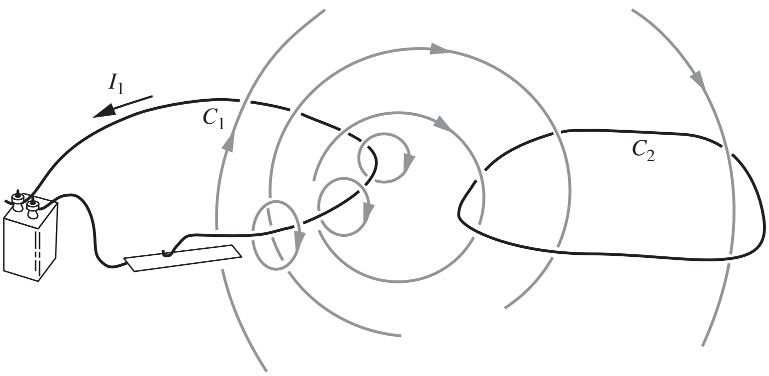

Figure 7.19.
Current I_1 in loop C_1 causes a certain flux Φ_{21} through loop C_2.

units, with $\mathcal{E}$ in volts and I in amperes, the unit for M_{21} is volt · amp^{-1} · s, or ohm · s. This unit is called the *henry*;[2]

$$1 \text{ henry} = 1 \, \frac{\text{volt} \cdot \text{second}}{\text{amp}} = 1 \text{ ohm} \cdot \text{second}. \tag{7.39}$$

That is, the mutual inductance M_{21} is one henry if a current I_1 changing at the rate of 1 ampere/second induces an electromotive force of 1 volt in circuit C_2. In Gaussian units, with $\mathcal{E}$ in statvolts and I in esu/second, the unit for M_{21} is statvolt · (esu/second)$^{-1}$ · second. Since 1 statvolt equals 1 esu/cm, this unit can also be written as second^2/cm.

Example (Concentric rings) Figure 7.20 shows two coplanar, concentric rings: a small ring C_2 and a much larger ring C_1. Assuming $R_2 \ll R_1$, what is the mutual inductance M_{21}?

Solution At the center of C_1, with I_1 flowing, the field B_1 is given by Eq. (6.54) as

$$B_1 = \frac{\mu_0 I_1}{2R_1}. \tag{7.40}$$

Since we are assuming $R_2 \ll R_1$, we can neglect the variation of B_1 over the interior of the small ring. The flux through the small ring is then

$$\Phi_{21} = (\pi R_2^2)\frac{\mu_0 I_1}{2R_1} = \frac{\mu_0 \pi I_1 R_2^2}{2R_1}. \tag{7.41}$$

The mutual inductance M_{21} in Eq. (7.37) is therefore

$$M_{21} = \frac{\Phi_{21}}{I_1} = \frac{\mu_0 \pi R_2^2}{2R_1}, \tag{7.42}$$

and the electromotive force induced in C_2 is

$$\mathcal{E}_{21} = -M_{21}\frac{dI_1}{dt} = -\frac{\mu_0 \pi R_2^2}{2R_1}\frac{dI_1}{dt}. \tag{7.43}$$

Since $\mu_0 = 4\pi \cdot 10^{-7}\,\text{kg m/C}^2$, we can write M_{21} alternatively as

$$M_{21} = \frac{(2\pi^2 \cdot 10^{-7}\,\text{kg m/C}^2)R_2^2}{R_1}. \tag{7.44}$$

The numerical value of this expression gives M_{21} in henrys. In Gaussian units, you can show that the relation corresponding to Eq. (7.43) is

$$\mathcal{E}_{21} = -\frac{1}{c}\frac{2\pi^2 R_2^2}{cR_1}\frac{dI_1}{dt}, \tag{7.45}$$

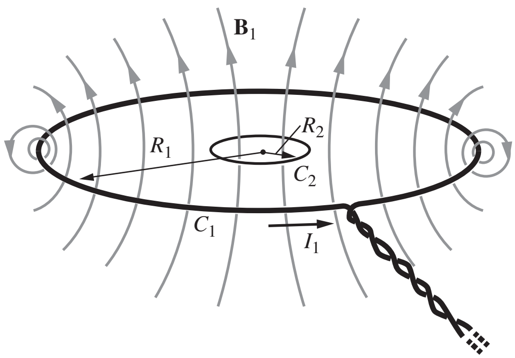

Figure 7.20.
Current I_1 in ring C_1 causes field $\mathbf{B}_1$, which is approximately uniform over the region of the small ring C_2.

[2] The unit is named after Joseph Henry (1797–1878), the foremost American physicist of his time. Electromagnetic induction was discovered independently by Henry, practically at the same time as Faraday conducted his experiments. Henry was the first to recognize the phenomenon of self-induction. He developed the electromagnet and the prototype of the electric motor, invented the electric relay, and all but invented telegraphy.

with $\mathcal{E}_{21}$ in statvolts, the R's in cm, and I_1 in esu/second. M_{21} is the coefficient of the dI_1/dt term, namely $2\pi^2 R_2^2/c^2 R_1$ (in second^2/cm). Appendix C states, and derives, the conversion factor from henry to second^2/cm.

Incidentally, the minus sign we have been carrying along doesn't tell us much at this stage. If you want to be sure which way the electromotive force will tend to drive current in C_2, Lenz's law is your most reliable guide.

If the circuit C_1 consisted of N_1 turns of wire instead of a single ring, the field B_1 at the center would be N_1 times as strong, for a given current I_1. Also, if the small loop C_2 consisted of N_2 turns, all of the same radius R_2, the electromotive force in each turn would add to that in the next, making the total electromotive force in that circuit N_2 times that of a single turn. Thus for *multiple turns* in each coil the mutual inductance will be given by

$$M_{21} = \frac{\mu_0 \pi N_1 N_2 R_2^2}{2R_1}. \tag{7.46}$$

This assumes that the turns in each coil are neatly bundled together, the cross section of the bundle being small compared with the coil radius. However, the mutual inductance M_{21} has a well-defined meaning for two circuits of any shape or distribution. As we wrote in Eq. (7.38), M_{21} is the (negative) ratio of the electromotive force in circuit 2, caused by changing current in circuit 1, to the rate of change of current I_1. That is,

$$M_{21} = -\frac{\mathcal{E}_{21}}{dI_1/dt}. \tag{7.47}$$

7.7 A reciprocity theorem

In considering the circuits C_1 and C_2 in the preceding example, we might have inquired about the electromotive force induced in circuit C_1 by a changing current in circuit C_2. That would involve another coefficient of mutual inductance, M_{12}, given by (ignoring the sign)

$$M_{12} = \frac{\mathcal{E}_{12}}{dI_2/dt}. \tag{7.48}$$

M_{12} is related to M_{21} by the following remarkable theorem.

Theorem 7.2 *For any two circuits,*

$$\boxed{M_{12} = M_{21}} \tag{7.49}$$

This theorem is not a matter of geometrical symmetry. Even the simple example in Fig. 7.20 is not symmetrical with respect to the two circuits. Note that R_1 and R_2 enter in different ways into the expression for M_{21}; Eq. (7.49) asserts that, for these two dissimilar circuits, if

$$M_{21} = \frac{\pi \mu_0 N_1 N_2 R_2^2}{2R_1}, \qquad \text{then} \qquad M_{12} = \frac{\pi \mu_0 N_1 N_2 R_2^2}{2R_1} \tag{7.50}$$

also – and *not* what we would get by switching 1's and 2's everywhere!

Proof In view of the definition of mutual inductance in Eq. (7.37), our goal is to show that $\Phi_{12}/I_2 = \Phi_{21}/I_1$, where Φ_{12} is the flux through some circuit C_1 due to a current I_2 in another circuit C_2, and Φ_{21} is the flux through C_2 due to a current I_1 in C_1. We will use the vector potential. Stokes' theorem tells us that

$$\int_C \mathbf{A} \cdot d\mathbf{s} = \int_S (\text{curl}\,\mathbf{A}) \cdot d\mathbf{a}. \tag{7.51}$$

In particular, if $\mathbf{A}$ is the vector potential of a magnetic field $\mathbf{B}$, in other words, if $\mathbf{B} = \text{curl}\,\mathbf{A}$, then we have

$$\boxed{\int_C \mathbf{A} \cdot d\mathbf{s} = \int_S \mathbf{B} \cdot d\mathbf{a} = \Phi_S} \tag{7.52}$$

That is, the line integral of the vector potential around a loop is equal to the flux of **B** *through the loop.*

Now, the vector potential is related to its current source as follows, according to Eq. (6.46):

$$\mathbf{A}_{21} = \frac{\mu_0 I_1}{4\pi} \int_{C_1} \frac{d\mathbf{s}_1}{r_{21}}, \tag{7.53}$$

where $\mathbf{A}_{21}$ is the vector potential, at some point (x_2, y_2, z_2), of the magnetic field caused by current I_1 flowing in circuit C_1; $d\mathbf{s}_1$ is an element of the loop C_1; and r_{21} is the magnitude of the distance from that element to the point (x_2, y_2, z_2).

Figure 7.21 shows the two loops C_1 and C_2, with current I_1 flowing in C_1. Let (x_2, y_2, z_2) be a point on the loop C_2. Then Eqs. (7.52) and (7.53) give the flux through C_2 due to current I_1 in C_1 as

$$\begin{aligned} \Phi_{21} &= \int_{C_2} d\mathbf{s}_2 \cdot \mathbf{A}_{21} = \int_{C_2} d\mathbf{s}_2 \cdot \frac{\mu_0 I_1}{4\pi} \int_{C_1} \frac{d\mathbf{s}_1}{r_{21}} \\ &= \frac{\mu_0 I_1}{4\pi} \int_{C_2} \int_{C_1} \frac{d\mathbf{s}_2 \cdot d\mathbf{s}_1}{r_{21}}. \end{aligned} \tag{7.54}$$

Similarly, the flux through C_1 due to current I_2 flowing in C_2 is given by the same expression with the labels 1 and 2 reversed:

$$\Phi_{12} = \frac{\mu_0 I_2}{4\pi} \int_{C_1} \int_{C_2} \frac{d\mathbf{s}_1 \cdot d\mathbf{s}_2}{r_{12}}. \tag{7.55}$$

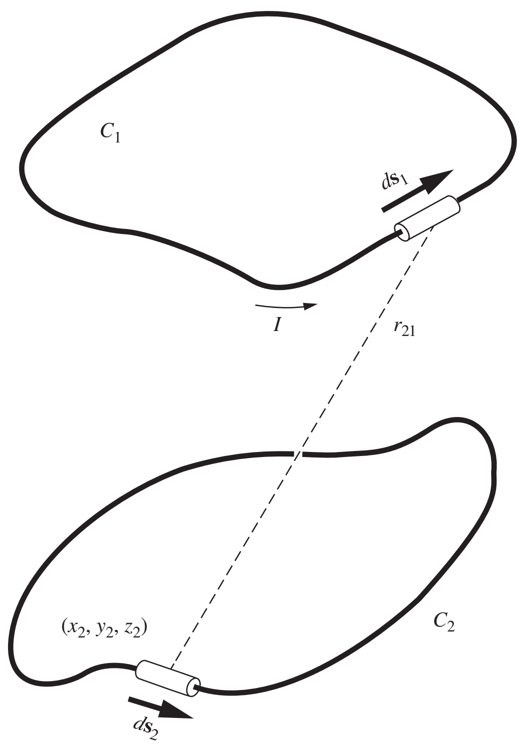

Figure 7.21.
Calculation of the flux Φ_{21} that passes through C_2 as a result of current I_1 flowing in C_1.

Now $r_{12} = r_{21}$, for these are just distance magnitudes, not vectors. The meaning of each of the integrals above is as follows: take the scalar product of a pair of line elements, one on each loop, divide by the distance between them, and sum over all pairs. The only difference between Eqs. (7.54) and (7.55) is the *order* in which this operation is carried out, and that cannot affect the final sum. Hence $\Phi_{21}/I_1 = \Phi_{12}/I_2$, as desired. Thanks to this theorem, we need make no distinction between M_{12} and M_{21}. We may speak, henceforth, of *the* mutual inductance M of any two circuits. □

Theorems of this sort are often called "reciprocity" theorems. There are some other reciprocity theorems on electric circuits not unrelated to this one. This may remind you of the relation $C_{jk} = C_{kj}$ mentioned in Section 3.6 and treated in Exercise 3.64. (In the spirit of that exercise, see Problem 7.10 for a second proof of the above $M_{12} = M_{21}$ theorem.) A reciprocity relation usually expresses some general symmetry law that is *not* apparent in the superficial structure of the system.

7.8 Self-inductance

When the current I_1 is changing, there is a change in the flux through circuit C_1 itself, and consequently an electromotive force is induced. Call this $\mathcal{E}_{11}$. The induction law holds, whatever the source of the flux:

$$\mathcal{E}_{11} = -\frac{d\Phi_{11}}{dt}, \tag{7.56}$$

where Φ_{11} is the flux through circuit 1 of the field B_1 due to the current I_1 in circuit 1. The minus sign expresses the fact that the electromotive force is always directed so as to *oppose* the *change* in current – Lenz's law, again. Since Φ_{11} will be proportional to I_1 we can write

$$\frac{\Phi_{11}}{I_1} = \text{constant} \equiv L_1. \tag{7.57}$$

Equation (7.56) then becomes

$$\mathcal{E}_{11} = -L_1\frac{dI_1}{dt}. \tag{7.58}$$

The constant L_1 is called the *self-inductance* of the circuit. We usually drop the subscript "1."

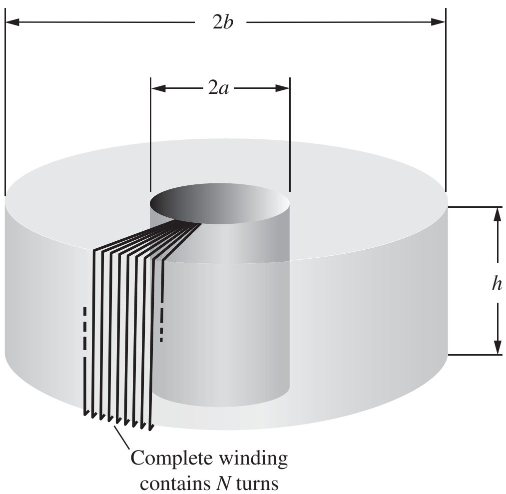

Figure 7.22.
Toroidal coil of rectangular cross section. Only a few turns are shown.

Example (Rectangular toroidal coil) As an example of a circuit for which L can be calculated, consider the rectangular toroidal coil of Exercise 6.61, shown here again in Fig. 7.22. You found (if you worked that exercise) that a current I flowing in the coil of N turns produces a field, the strength of which, at a radial distance r from the axis of the coil, is given by $B = \mu_0 NI/2\pi r$. The total flux through one turn of the coil is the integral of this field over the cross section of the coil:

$$\Phi(\text{one turn}) = h\int_a^b \frac{\mu_0 NI}{2\pi r}\,dr = \frac{\mu_0 NIh}{2\pi}\ln\left(\frac{b}{a}\right). \tag{7.59}$$

The flux threading the circuit of N turns is N times as great:

$$\Phi = \frac{\mu_0 N^2 Ih}{2\pi}\ln\left(\frac{b}{a}\right). \tag{7.60}$$

Hence the induced electromotive force $\mathcal{E}$ is

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{\mu_0 N^2 h}{2\pi}\ln\left(\frac{b}{a}\right)\frac{dI}{dt}. \tag{7.61}$$

Thus the self-inductance of this coil is given by

$$L = \frac{\mu_0 N^2 h}{2\pi}\ln\left(\frac{b}{a}\right). \tag{7.62}$$

Since $\mu_0 = 4\pi \cdot 10^{-7}\,\text{kg m/C}^2$, we can rewrite this in a form similar to Eq. (7.44):

$$L = (2\cdot 10^{-7}\,\text{kg m/C}^2)N^2 h\ln\left(\frac{b}{a}\right). \tag{7.63}$$

The numerical value of this expression gives L in henrys. In Gaussian units, you can show that the self-inductance is

$$L = \frac{2N^2 h}{c^2}\ln\left(\frac{b}{a}\right). \tag{7.64}$$

You may think that one of the rings we considered earlier would have made a simpler example to illustrate the calculation of self-inductance. However, if we try to calculate the inductance of a simple circular loop of wire, we encounter a puzzling difficulty. It seems a good idea to simplify the problem by assuming that the wire has zero diameter. But we soon discover that, if finite current flows in a filament of zero diameter, the flux threading a loop made of such a filament is infinite! The reason is that the field B, in the neighborhood of a filamentary current, varies as $1/r$, where r is the distance from the filament, and the integral of $B \times$ (area) diverges as $\int (dr/r)$ when we extend it down to $r = 0$. To avoid this we may let the radius of the wire be finite, not zero, which is more realistic anyway. This may make the calculation a bit more complicated, in a given case, but that won't worry us. The real difficulty is that different parts of the wire (at different distances from the center of the loop) now appear as *different circuits,* linked by different amounts of flux. We are no longer sure what we mean by *the* flux through *the* circuit. In fact, because the electromotive force is different in the different filamentary loops into which the circuit can be divided, some *redistribution* of current density must occur when rapidly changing currents flow in the ring. Hence the inductance of the circuit may depend somewhat on the rapidity of change of I, and thus not be strictly a constant as Eq. (7.58) would imply.

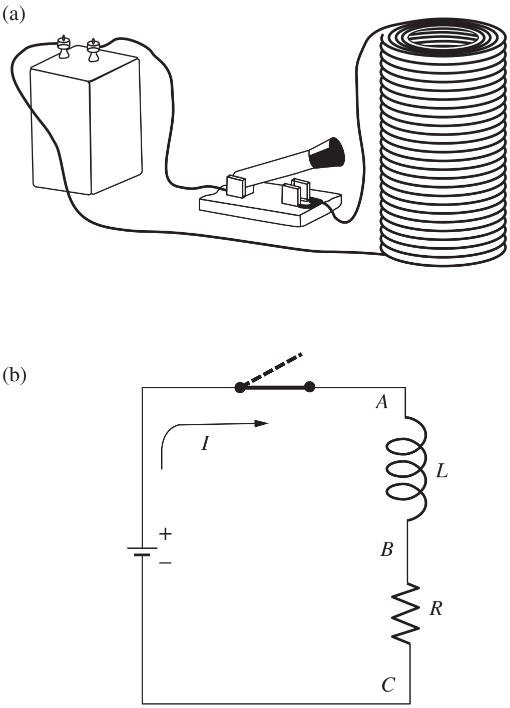

Figure 7.23.
A simple circuit with inductance (a) and resistance (b).

We avoided this embarrassment in the toroidal coil example by ignoring the field in the immediate vicinity of the individual turns of the winding. Most of the flux does *not* pass through the wires themselves, and whenever that is the case the effect we have just been worrying about will be unimportant.

7.9 Circuit containing self-inductance

Suppose we connect a battery, providing electromotive force $\mathcal{E}_0$, to a coil, or *inductor,* with self-inductance L, as in Fig. 7.23(a). The coil itself, the connecting wires, and even the battery will have some resistance. We don't care how this is distributed around the circuit. It can all be lumped together in one resistance R, indicated on the circuit diagram of Fig. 7.23(b) by a resistor symbol with this value. Also, the rest of the circuit, especially the connecting wires, contribute a bit to the self-inductance of the whole circuit; we assume that this is included in L. In other words, Fig. 7.23(b) represents an idealization of the physical circuit. The inductor L, symbolized by -ℓℓℓ-, has no resistance; the resistor R has no inductance. It is this idealized circuit that we shall now analyze.

If the current I in the circuit is changing at the rate dI/dt, an electromotive force $L\,dI/dt$ will be induced, in a direction to oppose the change. Also, there is the constant electromotive force $\mathcal{E}_0$ of the battery. If we define the positive current direction as the one in which the battery tends to drive current around the circuit, then the net electromotive force at any instant is $\mathcal{E}_0 - L\,dI/dt$. This drives the current I through the resistor R. That is,

$$\mathcal{E}_0 - L\frac{dI}{dt} = RI. \tag{7.65}$$

We can also describe the situation in this way: the voltage difference between points A and B in Fig. 7.23(b), which we call the *voltage across the inductor,* is $L\,dI/dt$, with the upper end of the inductor positive if I in the direction shown is *increasing*. The voltage difference between B and C, the voltage across the resistor, is RI, with the upper end of the resistor positive. Hence the sum of the voltage across the inductor and the voltage across the resistor is $L\,dI/dt + RI$. This is the same as the potential difference between the battery terminals, which is $\mathcal{E}_0$ (our idealized battery has no internal resistance). Thus we have

$$\mathcal{E}_0 = L\frac{dI}{dt} + RI, \tag{7.66}$$

which is merely a restatement of Eq. (7.65).

Before we look at the mathematical solution of Eq. (7.65), let's predict what ought to happen in this circuit if the switch is closed at $t = 0$. Before the switch is closed, $I = 0$, necessarily. A long time after the switch has been closed, some steady state will have been attained, with current practically constant at some value I_0. Then and thereafter, $dI/dt \approx 0$, and Eq. (7.65) reduces to

$$\mathcal{E}_0 = RI_0. \tag{7.67}$$

The transition from zero current to the steady-state current I_0 cannot occur abruptly at $t = 0$, for then dI/dt would be infinite. In fact, just after $t = 0$, the current I will be so small that the RI term in Eq. (7.65) can be ignored, giving

$$\frac{dI}{dt} = \frac{\mathcal{E}_0}{L}. \tag{7.68}$$

The inductance L limits the rate of rise of the current.

What we now know is summarized in Fig. 7.24(a). It only remains to find how the whole change takes place. Equation (7.65) is a differential equation very much like Eq. (4.39) in Chapter 4. The constant $\mathcal{E}_0$ term complicates things slightly, but the equation is still straightforward to solve. In Problem 7.14 you can show that the solution to Eq. (7.65) that satisfies our initial condition, $I = 0$ at $t = 0$, is

$$I(t) = \frac{\mathcal{E}_0}{R}\left(1 - e^{-(R/L)t}\right). \tag{7.69}$$

The graph in Fig. 7.24(b) shows the current approaching its asymptotic value I_0 exponentially. The "time constant" of this circuit is the quantity L/R. If L is measured in henrys and R in ohms, this comes out in seconds, since henrys = volt · amp^{-1} · second, and ohms = volt · amp^{-1}.

What happens if we open the switch after the current I_0 has been established, thus forcing the current to drop abruptly to zero? That would make the term $L\,dI/dt$ negatively infinite! The catastrophe can be more than mathematical. People have been killed opening switches in highly inductive circuits. What happens generally is that a very high induced voltage causes a spark or arc across the open switch contacts, so that the current continues after all. Let us instead remove the battery from the circuit by closing a conducting path *across* the LR combination, as in Fig. 7.25(a), at the same time disconnecting the battery. We now have a circuit described by the equation

$$0 = L\frac{dI}{dt} + RI, \tag{7.70}$$

with the initial condition $I = I_0$ at $t = t_1$, where t_1 is the instant at which the short circuit was closed. The solution is the simple exponential decay function

$$I(t) = I_0 e^{-(R/L)(t-t_1)} \tag{7.71}$$

with the same characteristic time L/R as before.

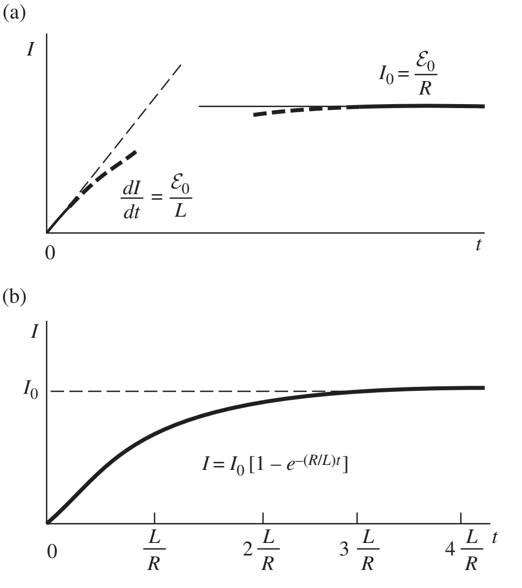

Figure 7.24.
(a) How the current must behave initially, and after a very long time has elapsed. (b) The complete variation of current with time in the circuit of Fig. 7.23.

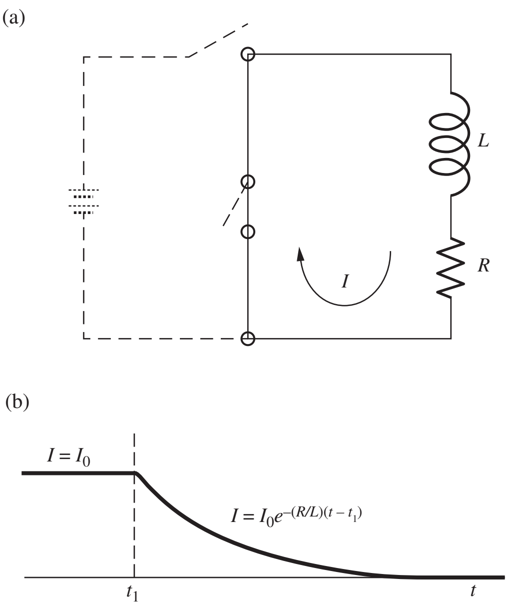

Figure 7.25.
(a) *LR* circuit. (b) Exponential decay of current in the *LR* circuit.

7.10 Energy stored in the magnetic field

During the decay of the current described by Eq. (7.71) and Fig. 7.25(b), energy is dissipated in the resistor R. Since the energy dU dissipated in any short interval dt is $RI^2\,dt$, the total energy dissipated after the closing of the switch at time t_1 is given by

$$U = \int_{t_1}^{\infty} RI^2\,dt = \int_{t_1}^{\infty} RI_0^2 e^{-(2R/L)(t-t_1)}\,dt$$
$$= -RI_0^2\left(\frac{L}{2R}\right)e^{-(2R/L)(t-t_1)}\Big|_{t_1}^{\infty} = \frac{1}{2}LI_0^2. \qquad (7.72)$$

The source of this energy was the inductor with its magnetic field. Indeed, exactly that amount of work had been done by the battery to build up the current in the first place – over and above the energy dissipated in the resistor between $t = 0$ and $t = t_1$, which was also provided by the battery. To see that this is a general relation, note that, if we have an increasing current in an inductor, work must be done to drive the current I against the induced electromotive force $L\,dI/dt$. Since the electromotive force is defined to be the work done per unit charge, and since a charge $I\,dt$ moves through the inductor in time dt, the work done in time dt is

$$dW = L\frac{dI}{dt}(I\,dt) = LI\,dI = \frac{1}{2}L\,d(I^2). \qquad (7.73)$$

Therefore, we may assign a total energy

$$\boxed{U = \frac{1}{2}LI^2} \qquad (7.74)$$

to an inductor carrying current I. With the eventual decay of this current, that amount of energy will appear somewhere else.

It is natural to regard this as energy stored in the magnetic field of the inductor, just as we have described the energy of a charged capacitor as stored in its electric field. The energy of a capacitor charged to potential difference V is $(1/2)CV^2$ and is accounted for by assigning to an element of volume dv, where the electric field strength is E, an amount of energy $(\epsilon_0/2)E^2\,dv$. It is pleasant, but hardly surprising, to find that a similar relation holds for the energy stored in an inductor. That is, we can ascribe to the magnetic field an energy density $(1/2\mu_0)B^2$, and summing the energy of the whole field will give the energy $(1/2)LI^2$.

Example (Rectangular toroidal coil) To show how the energy density $B^2/2\mu_0$ works out in one case, we can go back to the toroidal coil whose inductance L we calculated in Section 7.8. We found in Eq. (7.62) that

$$L = \frac{\mu_0 N^2 h}{2\pi}\ln\left(\frac{b}{a}\right). \qquad (7.75)$$

The magnetic field strength B, with current I flowing, was given by

$$B = \frac{\mu_0 NI}{2\pi r}. \tag{7.76}$$

To calculate the volume integral of $B^2/2\mu_0$ we can use a volume element consisting of the cylindrical shell sketched in Fig. 7.26, with volume $2\pi rh\,dr$. As this shell expands from $r = a$ to $r = b$, it sweeps through all the space that contains magnetic field. (The field B is zero everywhere outside the torus, remember.) So,

$$\frac{1}{2\mu_0}\int B^2\,dv = \frac{1}{2\mu_0}\int_a^b \left(\frac{\mu_0 NI}{2\pi r}\right)^2 2\pi rh\,dr = \frac{\mu_0 N^2 h I^2}{4\pi}\ln\left(\frac{b}{a}\right). \tag{7.77}$$

Comparing this result with Eq. (7.75), we see that, indeed,

$$\frac{1}{2\mu_0}\int B^2\,dv = \frac{1}{2}LI^2. \tag{7.78}$$

The task of Problem 7.18 is to show that this result holds for an arbitrary circuit with inductance L.

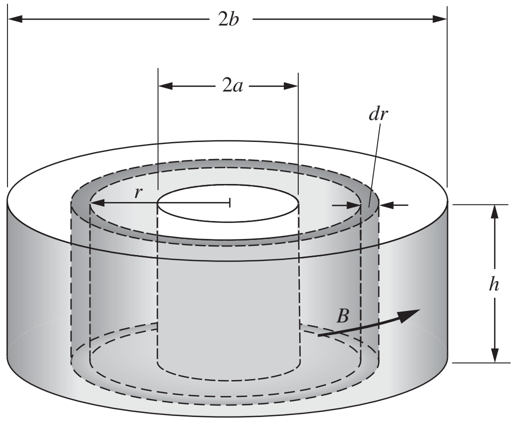

Figure 7.26.
Calculation of energy stored in the magnetic field of the toroidal coil of Fig. 7.22.

The more general statement, the counterpart of our statement for the electric field in Eq. (1.53), is that the energy U to be associated with any magnetic field $B(x, y, z)$ is given by

$$\boxed{U = \frac{1}{2\mu_0}\int_{\text{entire field}} B^2\,dv} \tag{7.79}$$

With B in tesla and v in m^3, the energy U will be given in joules, as you can check. In Eq. (7.74), with L in henrys and I in amperes, U will also be given in joules. The Gaussian equivalent of Eq. (7.79) for U in ergs, B in gauss, and v in cm^3 is

$$U = \frac{1}{8\pi}\int_{\text{entire field}} B^2\,dv. \tag{7.80}$$

The Gaussian equivalent of Eq. (7.74) remains $U = LI^2/2$, because the reasoning leading up to that equation is unchanged.

7.2.3 ■ Inductance

Suppose you have two loops of wire, at rest (Fig. 7.30). If you run a steady current I_1 around loop 1, it produces a magnetic field $\mathbf{B}_1$. Some of the field lines pass

[16]This paradox was suggested by Tom Colbert. Refer to Problem 2.55.

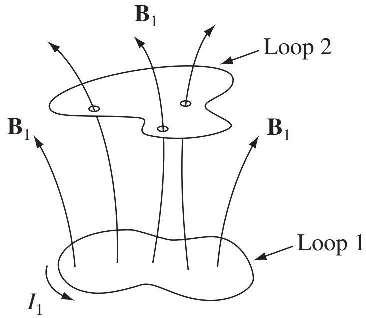

FIGURE 7.30

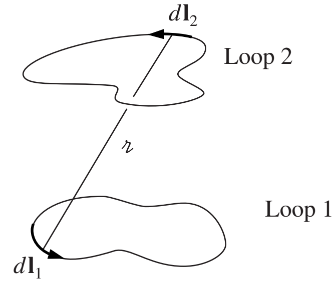

FIGURE 7.31

through loop 2; let Φ_2 be the flux of $\mathbf{B}_1$ through 2. You might have a tough time actually *calculating* $\mathbf{B}_1$, but a glance at the Biot-Savart law,

$$\mathbf{B}_1 = \frac{\mu_0}{4\pi} I_1 \oint \frac{d\mathbf{l}_1 \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2},$$

reveals one significant fact about this field: *It is proportional to the current* I_1. Therefore, so too is the flux through loop 2:

$$\Phi_2 = \int \mathbf{B}_1 \cdot d\mathbf{a}_2.$$

Thus

$$\Phi_2 = M_{21} I_1, \tag{7.22}$$

where M_{21} is the constant of proportionality; it is known as the **mutual inductance** of the two loops.

There is a cute formula for the mutual inductance, which you can derive by expressing the flux in terms of the vector potential, and invoking Stokes' theorem:

$$\Phi_2 = \int \mathbf{B}_1 \cdot d\mathbf{a}_2 = \int (\nabla \times \mathbf{A}_1) \cdot d\mathbf{a}_2 = \oint \mathbf{A}_1 \cdot d\mathbf{l}_2.$$

Now, according to Eq. 5.66,

$$\mathbf{A}_1 = \frac{\mu_0 I_1}{4\pi} \oint \frac{d\mathbf{l}_1}{\mathscr{r}},$$

and hence

$$\Phi_2 = \frac{\mu_0 I_1}{4\pi} \oint \left(\oint \frac{d\mathbf{l}_1}{\mathscr{r}} \right) \cdot d\mathbf{l}_2.$$

Evidently

$$M_{21} = \frac{\mu_0}{4\pi} \oint \oint \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{\mathscr{r}}. \tag{7.23}$$

This is the **Neumann formula**; it involves a double line integral—one integration around loop 1, the other around loop 2 (Fig. 7.31). It's not very useful for practical calculations, but it does reveal two important things about mutual inductance:

1. M_{21} is a purely geometrical quantity, having to do with the sizes, shapes, and relative positions of the two loops.

2. The integral in Eq. 7.23 is unchanged if we switch the roles of loops 1 and 2; it follows that

$$M_{21} = M_{12}. \tag{7.24}$$

This is an astonishing conclusion: *Whatever the shapes and positions of the loops, the flux through 2 when we run a current I around 1 is identical to the flux through 1 when we send the same current I around 2*. We may as well drop the subscripts and call them both M.

Example 7.10. A short solenoid (length l and radius a, with n_1 turns per unit length) lies on the axis of a very long solenoid (radius b, n_2 turns per unit length) as shown in Fig. 7.32. Current I flows in the short solenoid. What is the flux through the long solenoid?

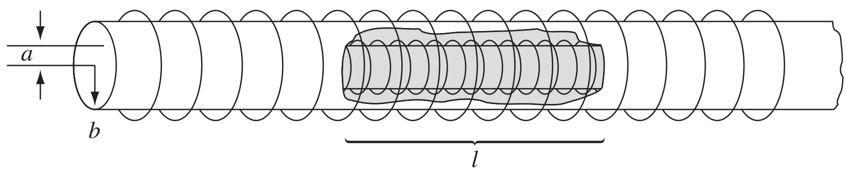

FIGURE 7.32

Solution

Since the inner solenoid is short, it has a very complicated field; moreover, it puts a different flux through each turn of the outer solenoid. It would be a *miserable* task to compute the total flux this way. However, if we exploit the equality of the mutual inductances, the problem becomes very easy. Just look at the reverse situation: run the current I through the *outer* solenoid, and calculate the flux through the *inner* one. The field inside the long solenoid is constant:

$$B = \mu_0 n_2 I$$

(Eq. 5.59), so the flux through a single loop of the short solenoid is

$$B\pi a^2 = \mu_0 n_2 I \pi a^2.$$

There are $n_1 l$ turns in all, so the total flux through the inner solenoid is

$$\Phi = \mu_0 \pi a^2 n_1 n_2 l I.$$

This is also the flux a current I in the *short* solenoid would put through the *long* one, which is what we set out to find. Incidentally, the mutual inductance, in this case, is

$$M = \mu_0 \pi a^2 n_1 n_2 l.$$

Suppose, now, that you *vary* the current in loop 1. The flux through loop 2 will vary accordingly, and Faraday's law says this changing flux will induce an emf in loop 2:

$$\mathcal{E}_2 = -\frac{d\Phi_2}{dt} = -M\frac{dI_1}{dt}. \tag{7.25}$$

(In quoting Eq. 7.22—which was based on the Biot-Savart law—I am tacitly assuming that the currents change slowly enough for the system to be considered quasistatic.) What a remarkable thing: Every time you change the current in loop 1, an induced current flows in loop 2—even though there are no wires connecting them!

Come to think of it, a changing current not only induces an emf in any nearby loops, it also induces an emf in the source loop *itself* (Fig 7.33). Once again, the field (and therefore also the flux) is proportional to the current:

$$\Phi = LI. \tag{7.26}$$

The constant of proportionality L is called the **self inductance** (or simply the **inductance**) of the loop. As with M, it depends on the geometry (size and shape) of the loop. If the current changes, the emf induced in the loop is

$$\mathcal{E} = -L\frac{dI}{dt}. \tag{7.27}$$

Inductance is measured in **henries** (H); a henry is a volt-second per ampere.

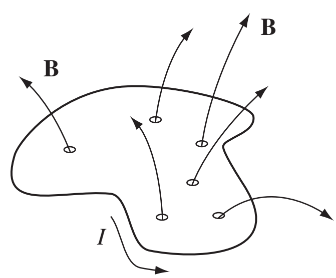

FIGURE 7.33

Example 7.11. Find the self-inductance of a toroidal coil with rectangular cross section (inner radius a, outer radius b, height h), that carries a total of N turns.

Solution

The magnetic field inside the toroid is (Eq. 5.60)

$$B = \frac{\mu_0 N I}{2\pi s}.$$

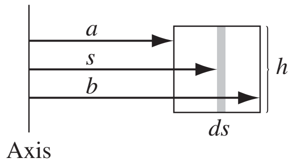

FIGURE 7.34

The flux through a single turn (Fig. 7.34) is

$$\int \mathbf{B} \cdot d\mathbf{a} = \frac{\mu_0 N I}{2\pi} h \int_a^b \frac{1}{s}\, ds = \frac{\mu_0 N I h}{2\pi} \ln\left(\frac{b}{a}\right).$$

The *total* flux is N times this, so the self-inductance (Eq. 7.26) is

$$L = \frac{\mu_0 N^2 h}{2\pi} \ln\left(\frac{b}{a}\right). \tag{7.28}$$

Inductance (like capacitance) is an intrinsically *positive* quantity. Lenz's law, which is enforced by the minus sign in Eq. 7.27, dictates that the emf is in such a direction as to *oppose* any *change in current*. For this reason, it is called a **back emf**. Whenever you try to alter the current in a wire, you must fight against this back emf. Inductance plays somewhat the same role in electric circuits that *mass* plays in mechanical systems: The greater L is, the harder it is to change the current, just as the larger the mass, the harder it is to change an object's velocity.

Example 7.12. Suppose a current I is flowing around a loop, when someone suddenly cuts the wire. The current drops "instantaneously" to zero. This generates a whopping back emf, for although I may be small, dI/dt is enormous. (That's why you sometimes draw a spark when you unplug an iron or toaster—electromagnetic induction is desperately trying to keep the current going, even if it has to jump the gap in the circuit.)

Nothing so dramatic occurs when you plug *in* a toaster or iron. In this case induction opposes the sudden *increase* in current, prescribing instead a smooth and continuous buildup. Suppose, for instance, that a battery (which supplies a constant emf $\mathcal{E}_0$) is connected to a circuit of resistance R and inductance L (Fig. 7.35). What current flows?

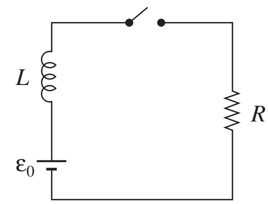

FIGURE 7.35

Solution

The total emf in this circuit is $\mathcal{E}_0$ from the battery plus $-L(dI/dt)$ from the inductance. Ohm's law, then, says[17]

$$\mathcal{E}_0 - L\frac{dI}{dt} = IR.$$

This is a first-order differential equation for I as a function of time. The general solution, as you can show for yourself, is

$$I(t) = \frac{\mathcal{E}_0}{R} + ke^{-(R/L)t},$$

where k is a constant to be determined by the initial conditions. In particular, if you close the switch at time $t = 0$, so $I(0) = 0$, then $k = -\mathcal{E}_0/R$, and

$$I(t) = \frac{\mathcal{E}_0}{R}\left[1 - e^{-(R/L)t}\right]. \tag{7.29}$$

This function is plotted in Fig. 7.36. Had there been no inductance in the circuit, the current would have jumped immediately to $\mathcal{E}_0/R$. In practice, *every* circuit has *some* self-inductance, and the current approaches $\mathcal{E}_0/R$ asymptotically. The quantity $\tau \equiv L/R$ is the **time constant**; it tells you how long the current takes to reach a substantial fraction (roughly two-thirds) of its final value.

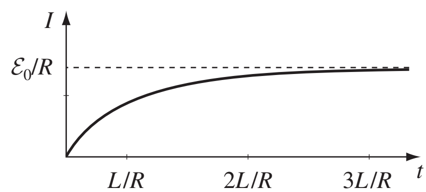

FIGURE 7.36

[17]Notice that $-L(dI/dt)$ goes on the *left* side of the equation—it is part of the emf that establishes the voltage across the resistor.

Problem 7.22 A small loop of wire (radius a) is held a distance z above the center of a large loop (radius b), as shown in Fig. 7.37. The planes of the two loops are parallel, and perpendicular to the common axis.

(a) Suppose current I flows in the big loop. Find the flux through the little loop. (The little loop is so small that you may consider the field of the big loop to be essentially constant.)

(b) Suppose current I flows in the little loop. Find the flux through the big loop. (The little loop is so small that you may treat it as a magnetic dipole.)

(c) Find the mutual inductances, and confirm that $M_{12} = M_{21}$.

Problem 7.23 A square loop of wire, of side a, lies midway between two long wires, $3a$ apart, and in the same plane. (Actually, the long wires are sides of a large rectangular loop, but the short ends are so far away that they can be neglected.) A clockwise current I in the square loop is gradually increasing: $dI/dt = k$ (a constant). Find the emf induced in the big loop. Which way will the induced current flow?

Problem 7.24 Find the self-inductance per unit length of a long solenoid, of radius R, carrying n turns per unit length.

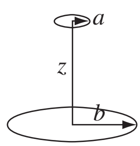

FIGURE 7.37

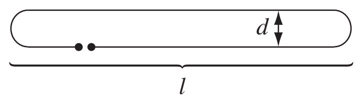

FIGURE 7.38

Problem 7.25 Try to compute the self-inductance of the "hairpin" loop shown in Fig. 7.38. (Neglect the contribution from the ends; most of the flux comes from the long straight section.) You'll run into a snag that is characteristic of many self-inductance calculations. To get a definite answer, assume the wire has a tiny radius ϵ, and ignore any flux through the wire itself.

Problem 7.26 An alternating current $I(t) = I_0 \cos(\omega t)$ (amplitude 0.5 A, frequency 60 Hz) flows down a straight wire, which runs along the axis of a toroidal coil with rectangular cross section (inner radius 1 cm, outer radius 2 cm, height 1 cm, 1000 turns). The coil is connected to a 500 Ω resistor.

(a) In the quasistatic approximation, what emf is induced in the toroid? Find the current, $I_R(t)$, in the resistor.

(b) Calculate the back emf in the coil, due to the current $I_R(t)$. What is the ratio of the amplitudes of this back emf and the "direct" emf in (a)?

Problem 7.27 A capacitor C is charged up to a voltage V and connected to an inductor L, as shown schematically in Fig. 7.39. At time $t = 0$, the switch S is closed. Find the current in the circuit as a function of time. How does your answer change if a resistor R is included in series with C and L?

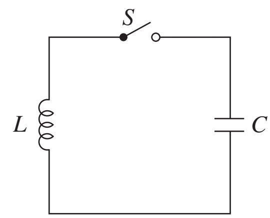

FIGURE 7.39

7.2.4 ■ Energy in Magnetic Fields

It takes a certain amount of energy to start a current flowing in a circuit. I'm not talking about the energy delivered to the resistors and converted into heat—that is irretrievably lost, as far as the circuit is concerned, and can be large or small, depending on how long you let the current run. What I am concerned with, rather, is the work you must do *against the back emf* to get the current going. This is a *fixed* amount, and it is *recoverable*: you get it back when the current is turned off. In the meantime, it represents energy latent in the circuit; as we'll see in a moment, it can be regarded as energy stored in the magnetic field.

The work done on a unit charge, against the back emf, in one trip around the circuit is $-\mathcal{E}$ (the minus sign records the fact that this is the work done *by you against* the emf, not the work done by the emf). The amount of charge per unit time passing down the wire is I. So the total work done per unit time is

$$\frac{dW}{dt} = -\mathcal{E} I = L I \frac{dI}{dt}.$$

If we start with zero current and build it up to a final value I, the work done (integrating the last equation over time) is

$$\boxed{W = \frac{1}{2} L I^2.} \tag{7.30}$$

It does not depend on how *long* we take to crank up the current, only on the geometry of the loop (in the form of L) and the final current I.

There is a nicer way to write W, which has the advantage that it is readily generalized to surface and volume currents. Remember that the flux Φ through the loop is equal to LI (Eq. 7.26). On the other hand,

$$\Phi = \int \mathbf{B} \cdot d\mathbf{a} = \int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l},$$

where the line integral is around the perimeter of the loop. Thus

$$LI = \oint \mathbf{A} \cdot d\mathbf{l},$$

and therefore

$$W = \frac{1}{2} I \oint \mathbf{A} \cdot d\mathbf{l} = \frac{1}{2} \oint (\mathbf{A} \cdot \mathbf{I})\, dl. \tag{7.31}$$

In this form, the generalization to volume currents is obvious:

$$W = \frac{1}{2} \int_{\mathcal{V}} (\mathbf{A} \cdot \mathbf{J})\, d\tau. \tag{7.32}$$

But we can do even better, and express W entirely in terms of the magnetic field: Ampère's law, $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$, lets us eliminate $\mathbf{J}$:

$$W = \frac{1}{2\mu_0} \int \mathbf{A} \cdot (\nabla \times \mathbf{B})\, d\tau. \tag{7.33}$$

Integration by parts transfers the derivative from $\mathbf{B}$ to $\mathbf{A}$; specifically, product rule 6 states that

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}),$$

so

$$\mathbf{A} \cdot (\nabla \times \mathbf{B}) = \mathbf{B} \cdot \mathbf{B} - \nabla \cdot (\mathbf{A} \times \mathbf{B}).$$

Consequently,

$$\begin{aligned} W &= \frac{1}{2\mu_0} \left[\int B^2\, d\tau - \int \nabla \cdot (\mathbf{A} \times \mathbf{B})\, d\tau \right] \\ &= \frac{1}{2\mu_0} \left[\int_{\mathcal{V}} B^2\, d\tau - \oint_{\mathcal{S}} (\mathbf{A} \times \mathbf{B}) \cdot d\mathbf{a} \right], \end{aligned} \tag{7.34}$$

where $\mathcal{S}$ is the surface bounding the volume $\mathcal{V}$.

Now, the integration in Eq. 7.32 is to be taken over the *entire volume occupied by the current*. But any region *larger* than this will do just as well, for $\mathbf{J}$ is zero out there anyway. In Eq. 7.34, the larger the region we pick the greater is the contribution from the volume integral, and therefore the smaller is that of the surface integral (this makes sense: as the surface gets farther from the current, both $\mathbf{A}$ and $\mathbf{B}$ decrease). In particular, if we agree to integrate over *all* space, then the surface integral goes to zero, and we are left with

$$\boxed{W = \frac{1}{2\mu_0} \int_{\text{all space}} B^2\, d\tau.} \tag{7.35}$$

In view of this result, we say the energy is "stored in the magnetic field," in the amount $(B^2/2\mu_0)$ per unit volume. This is a nice way to think of it, though someone looking at Eq. 7.32 might prefer to say that the energy is stored in the *current distribution*, in the amount $\frac{1}{2}(\mathbf{A} \cdot \mathbf{J})$ per unit volume. The distinction is one of bookkeeping; the important quantity is the total energy W, and we need not worry about where (if anywhere) the energy is "located."

You might find it strange that it takes energy to set up a magnetic field—after all, magnetic fields *themselves* do no work. The point is that producing a magnetic field, where previously there was none, requires *changing* the field, and a changing **B**-field, according to Faraday, induces an *electric* field. The latter, of course, *can* do work. In the beginning, there is no **E**, and at the end there is no **E**; but in between, while **B** is building up, there *is* an **E**, and it is against *this* that the work is done. (You see why I could not calculate the energy stored in a magnetostatic field back in Chapter 5.) In the light of this, it is extraordinary how similar the magnetic energy formulas are to their electrostatic counterparts:[18]

$$W_{\text{elec}} = \frac{1}{2}\int (V\rho)\,d\tau = \frac{\epsilon_0}{2}\int E^2\,d\tau, \qquad \text{(2.43 and 2.45)}$$

$$W_{\text{mag}} = \frac{1}{2}\int (\mathbf{A}\cdot\mathbf{J})\,d\tau = \frac{1}{2\mu_0}\int B^2\,d\tau. \qquad \text{(7.32 and 7.35)}$$

Example 7.13. A long coaxial cable carries current I (the current flows down the surface of the inner cylinder, radius a, and back along the outer cylinder, radius b) as shown in Fig. 7.40. Find the magnetic energy stored in a section of length l.

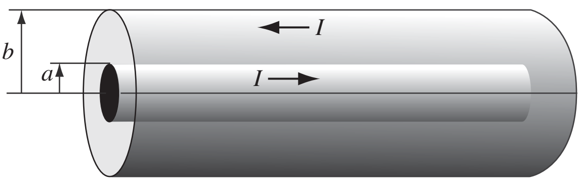

FIGURE 7.40

Solution

According to Ampère's law, the field between the cylinders is

$$\mathbf{B} = \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}}.$$

Elsewhere, the field is zero. Thus, the energy per unit volume is

$$\frac{1}{2\mu_0}\left(\frac{\mu_0 I}{2\pi s}\right)^2 = \frac{\mu_0 I^2}{8\pi^2 s^2}.$$

The energy in a cylindrical shell of length l, radius s, and thickness ds, then, is

$$\left(\frac{\mu_0 I^2}{8\pi^2 s^2}\right) 2\pi l s\,ds = \frac{\mu_0 I^2 l}{4\pi}\left(\frac{ds}{s}\right).$$

[18]For an illuminating confirmation of Eq. 7.35, using the method of Prob. 2.44, see T. H. Boyer, *Am. J. Phys.* **69**, 1 (2001).

Integrating from a to b, we have:

$$W = \frac{\mu_0 I^2 l}{4\pi} \ln\left(\frac{b}{a}\right).$$

By the way, this suggests a very simple way to calculate the self-inductance of the cable. According to Eq. 7.30, the energy can also be written as $\frac{1}{2}LI^2$. Comparing the two expressions,[19]

$$L = \frac{\mu_0 l}{2\pi} \ln\left(\frac{b}{a}\right).$$

This method of calculating self-inductance is especially useful when the current is not confined to a single path, but spreads over some surface or volume, so that different parts of the current enclose different amounts of flux. In such cases, it can be very tricky to get the inductance directly from Eq. 7.26, and it is best to let Eq. 7.30 *define* L.

Problem 7.28 Find the energy stored in a section of length l of a long solenoid (radius R, current I, n turns per unit length), (a) using Eq. 7.30 (you found L in Prob. 7.24); (b) using Eq. 7.31 (we worked out **A** in Ex. 5.12); (c) using Eq. 7.35; (d) using Eq. 7.34 (take as your volume the cylindrical tube from radius $a < R$ out to radius $b > R$).

Problem 7.29 Calculate the energy stored in the toroidal coil of Ex. 7.11, by applying Eq. 7.35. Use the answer to check Eq. 7.28.

Problem 7.30 A long cable carries current in one direction uniformly distributed over its (circular) cross section. The current returns along the surface (there is a very thin insulating sheath separating the currents). Find the self-inductance per unit length.

Problem 7.31 Suppose the circuit in Fig. 7.41 has been connected for a long time when suddenly, at time $t = 0$, switch S is thrown from A to B, bypassing the battery.

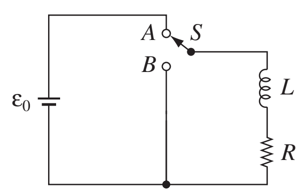

FIGURE 7.41

[19]Notice the similarity to Eq. 7.28—in a sense, the rectangular toroid *is* a short coaxial cable, turned on its side.

(a) What is the current at any subsequent time t?

(b) What is the total energy delivered to the resistor?

(c) Show that this is equal to the energy originally stored in the inductor.

Problem 7.32 Two tiny wire loops, with areas $\mathbf{a}_1$ and $\mathbf{a}_2$, are situated a displacement $\boldsymbol{\mathscr{r}}$ apart (Fig. 7.42).

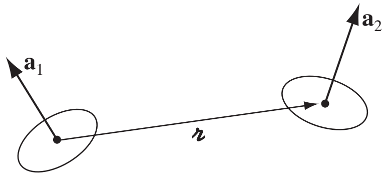

FIGURE 7.42

(a) Find their mutual inductance. [*Hint:* Treat them as magnetic dipoles, and use Eq. 5.88.] Is your formula consistent with Eq. 7.24?

(b) Suppose a current I_1 is flowing in loop 1, and we propose to turn on a current I_2 in loop 2. How much work must be done, against the mutually induced emf, to keep the current I_1 flowing in loop 1? In light of this result, comment on Eq. 6.35.

Problem 7.33 An infinite cylinder of radius R carries a uniform surface charge σ. We propose to set it spinning about its axis, at a final angular velocity ω_f. How much work will this take, per unit length? Do it two ways, and compare your answers:

(a) Find the magnetic field and the induced electric field (in the quasistatic approximation), inside and outside the cylinder, in terms of ω, $\dot{\omega}$, and s (the distance from the axis). Calculate the torque you must exert, and from that obtain the work done per unit length ($W = \int N\, d\phi$).

(b) Use Eq. 7.35 to determine the energy stored in the resulting magnetic field.